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Propagation of a quantum state in a continuum of coupled oscillators and applications to entanglement

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Abstract

The exponential Hilbert space approach is used to link quantum mechanics in a small system with d-dimensional Hilbert space $\mathcal{G}$, with the collective quantum behaviour in a large system comprised of d coupled oscillators with Hilbert space H. In the large system, the expectation value of the mode position operator $\mathcal{U}_x$ describes the location of a quantum state within the chain of d oscillators, and the expectation value of the mode momentum operator $\mathcal{U}_p$ describes the change of the mode position with time. A continuum of modes ($d \to \infty$) is also considered and in this case these operators obey the commutation relation $[\mathcal{U}_x, \mathcal{U}_p] = \mathrm{i}\mathcal{U}_{\mathbf{1}}$. A consequence of this is that uncertainties in the location of a state and in its momentum obey an uncertainty relation. Displacements and squeezing in the mode phase space are discussed. For a certain Hamiltonian, the $\mathcal{U}_x, \mathcal{U}_p$ obey equations of motion which are very similar to those of a harmonic oscillator. If the system is in an entangled state, the formalism can be used to quantify concepts like the location of entanglement, and the speed with which the entanglement propagates.

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1. Introduction

The exponential Hilbert space approach to the Fock space has been introduced in [1]. It has mainly been used for mathematical questions such as the construction of unitarily inequivalent representations. In this paper we use this approach in a more physical context. We show that it can provide a very interesting link between quantum phase space methods of a single oscillator and quantum techniques in a large system comprised of a continuum of oscillators.

In a recent paper [2] we have introduced the concept of mode phase space in a d-mode system, comprised of d coupled oscillators. We have defined mode-position and mode-momentum operators $\mathcal{U}_x$ and $\mathcal{U}_p$, which act collectively on all oscillators and have eigenvalues in $\mathcal{Z}_d$ (the integers modulo d). The expectation value of $\mathcal{U}_x$ is the 'average mode' in which the photons are located, and the expectation value of $\mathcal{U}_p$ shows how fast the mode position changes with time. The mode phase space is $\mathcal{Z}_d \times \mathcal{Z}_d$. Exponentials of $\mathcal{U}_x$ and $\mathcal{U}_p$ perform mode displacements of a quantum state in the mode phase space, i.e., they translate it along the chain of oscillators, and they also change its mode momentum. The mode displacements form a Heisenberg–Weyl group.

In this paper we first approach this work from a different angle. We show that the exponential Hilbert space formalism of [1] connects quantum techniques in a 'small' d-dimensional Hilbert space $\mathcal{G}$ with the collective quantum behaviour in a 'large' d-mode Hilbert space H which describes a system of d coupled oscillators. For example, displacements in $\mathcal{G}$ correspond to mode displacements in H. This formalism can be generalized to the continuum limit, where we go from the cyclic lattice of d oscillators to a continuum of coupled oscillators on a straight line (an infinite 'string').

The main part of the paper is the study of the continuum limit. In this case the index of the mode takes all real values, and the Hilbert space H is a *continuous tensor product* [3–5]. $\mathcal{G}$ becomes an infinite-dimensional harmonic oscillator Hilbert space, and the exponential Hilbert space formalism connects $\mathcal{G}$ with the Hilbert space H of the continuum of modes. We show that the mode position $\mathcal{U}_x$ and the mode momentum $\mathcal{U}_p$ obey the commutation relation $[\mathcal{U}_x, \mathcal{U}_p] = \mathrm{i}\mathcal{U}_{\mathbf{1}}$. Based on this we prove an uncertainty relation that involves the mode position and mode momentum. The quantum state of the system is shared by the oscillators labelled with x inside the mode uncertainty ellipse, and the rest of the oscillators are close to the vacuum state. Also the momenta with which the state propagates are those inside the mode uncertainty ellipse.

Displacements in the mode phase space, which is now $R \times R$, translate a state along the continuum of oscillators, and they also change its mode momentum. They are *collective* type of transformations, and they are very different from displacements within the phase space of a particular oscillator.

Mode squeezing transformations change the mode uncertainties. A quantum state, which is shared by the oscillators within the interval $(\langle\mathcal{U}_x\rangle - \Delta x, \langle\mathcal{U}_x\rangle + \Delta x)$, is transformed into another state shared by the oscillators in a smaller (or larger) interval, and at the same time the uncertainty in the mode momentum with which the state propagates increases (or decreases). They are also *collective* type of transformations, and they are very different from squeezing within the phase space of a particular oscillator.

The time evolution of the system is also studied. It is shown that for a certain Hamiltonian the mode position operator $\mathcal{U}_x$ and the mode momentum operator $\mathcal{U}_p$ obey equations of motion similar to those of a harmonic oscillator.

The formalism is applicable to all states; regardless of whether they are entangled or not. However if the states are entangled, the mode uncertainty ellipse can be used to quantify the location of entanglement and the speed with which entanglement propagates.

2. *H* as exponential Hilbert space

We consider a system comprised of d harmonic oscillators with Hilbert space $H = h \otimes \cdots \otimes h$. The creation and annihilation operators corresponding to the M mode are

$$\begin{aligned} &a_M^\dagger = \mathbf{1}_h \otimes \cdots \otimes a^\dagger \otimes \cdots \otimes \mathbf{1}_h, \qquad a_M = \mathbf{1}_h \otimes \cdots \otimes a \otimes \cdots \otimes \mathbf{1}_h, \\ &\left[a_M, a_K^\dagger\right] = \delta(M, K), \qquad M, K = 0, \ldots, (d-1). \end{aligned} \tag{1}$$

The indices belong in $\mathcal{Z}_d$ (the integers modulo d). $\delta(M, K)$ is Kronecker's delta; it is equal to 1 when $M = K$ (modulo d). The position and momentum of the Mth oscillator are

$$q_M = 2^{-1/2}\left[a_M^\dagger + a_M\right], \qquad r_M = 2^{-1/2}\mathrm{i}\left[a_M^\dagger - a_M\right]. \tag{2}$$

We also consider a d-dimensional Hilbert space $\mathcal{G}$ comprised of vectors $(z_0, z_1, \ldots, z_{d-1})$. For simplicity we use the short-hand notation z_M (with $M \in \mathcal{Z}_d$) for these vectors. Their scalar product is

$$(z_M, w_M) = \sum_{M=0}^{d-1} z_M^* w_M. \tag{3}$$

We introduce a map between vectors in $\mathcal{G}$ and coherent states in H as follows:

$$z_M \to |\{z_M\}\rangle_{\mathrm{coh}} \equiv |z_0\rangle_{\mathrm{coh}} \otimes \cdots \otimes |z_{d-1}\rangle_{\mathrm{coh}}. \tag{4}$$

It is easily seen that the scalar product of two coherent states can be written in terms of the scalar product of the corresponding vectors in $\mathcal{G}$:

$$[\mathcal{N}_z \mathcal{N}_w]^{-1}\, {}_{\mathrm{coh}}\langle\{z_M\}|\{w_M\}\rangle_{\mathrm{coh}} = \exp[(z_M, w_M)] \tag{5}$$

where $\mathcal{N}_z$ is the normalization coefficient,

$$\mathcal{N}_z = \exp\left[-\tfrac{1}{2}(z_M, z_M)\right]. \tag{6}$$

The total number of photons in the coherent state $|\{z_M\}\rangle_{\mathrm{coh}}$ is

$${}_{\mathrm{coh}}\langle\{z_M\}|n_T|\{z_M\}\rangle_{\mathrm{coh}} = (z_M, z_M). \tag{7}$$

It is seen that the normalization of the vectors z_M in $\mathcal{G}$ is important, because it is related to the total average number of photons in the corresponding coherent states in H. Therefore in the Hilbert space $\mathcal{G}$ we should consider not only vectors with length equal to one, but all vectors with all lengths. The coherent states $|\{z_M\}\rangle_{\mathrm{coh}}$ in H are of course normalized to 1.

We next introduce the Fock space defined by $\mathcal{G}$ which is the direct sum of all symmetric tensor products of $\mathcal{G}$

$$\mathrm{EXP}(\mathcal{G}) \equiv C \oplus \mathcal{G} \oplus (\mathcal{G} \otimes \mathcal{G})_{\mathrm{sym}} \oplus \cdots \tag{8}$$

In the space $\mathrm{EXP}(\mathcal{G})$ we introduce the 'exponential states'

$$\mathrm{EXP}[z_M] \equiv 1 \oplus z_M \oplus 2^{-1/2}(z_M \otimes z_M) \oplus \cdots \tag{9}$$

The first term 1 is the Fock vacuum. It is clear that we use the notation 'EXP' for the maps of equations (8), (9), and the notation 'exp' for the usual exponential.

The overlap of two exponential states is

$$\langle \mathrm{EXP}[z_M], \mathrm{EXP}[w_M]\rangle = \exp[(z_M, w_M)]. \tag{10}$$

Comparison of equation (5) with (10) shows that we can define a one-to-one map between $\mathrm{EXP}(\mathcal{G})$ and H if we identify the exponential states with the coherent states (with appropriate normalization):

$$|\{z_M\}\rangle_{\mathrm{coh}} = \mathcal{N}_z\, \mathrm{EXP}[z_M]. \tag{11}$$

Since the coherent states form an overcomplete set of states, we conclude that the spaces $\mathrm{EXP}(\mathcal{G})$ and H are isomorphic and write $H = \mathrm{EXP}(\mathcal{G})$.

2.1. General states

The 'EXP' map of equation (9) maps all states in $\mathcal{G}$ into a subset of $H = \mathrm{EXP}(\mathcal{G})$ comprised of the coherent states only. Physically, the vectors in $\mathcal{G}$ describe the distribution of the amplitude of the coherent states in H in the various modes.

Superpositions of vectors in $\mathcal{G}$ give another vector in $\mathcal{G}$ to which corresponds a single coherent state in H. Superpositions of coherent states in H give all pure states in H. Using the resolution of the identity for coherent states we show that an arbitrary pure state $|s\rangle$ in H can be written as

$$|s\rangle = \int \mathcal{D}^2 z s(\{z_M\})|\{z_M^*\}\rangle_{\mathrm{coh}}, \qquad s(\{z_M\}) =_{\mathrm{coh}} \langle\{z_M^*\}|s\rangle, \qquad \mathcal{D}^2 z \equiv \prod_{M\in\mathcal{Z}_d} \frac{d^2 z_M}{\pi}. \tag{12}$$

2.2. General operators

To a given operator θ acting on $\mathcal{G}$ we associate the operator $\mathrm{EXP}(\theta)$ acting on H defined as follows:

$$\mathrm{EXP}(\theta) \equiv \mathbf{1}_{\mathcal{G}} \oplus \theta \oplus (\theta \otimes \theta) \oplus \cdots. \tag{13}$$

It is easily seen that

$$\mathrm{EXP}(\theta)|\{z_M\}\rangle = \mathcal{N}_z\, \mathrm{EXP}(\theta)\, \mathrm{EXP}[z_M] = \mathcal{N}_z\, \mathrm{EXP}\,[\theta z_M] = \mathcal{N}_z \mathcal{N}_{\theta z}^{-1}|\theta\{z_M\}\rangle. \tag{14}$$

The 'EXP' map of equation (13) maps all the operators θ acting on G into a small subset within the set of all operators acting on H. We can easily show

$$\mathrm{EXP}(\mathbf{1}_{\mathcal{G}}) = \mathbf{1}_H, \qquad \mathrm{EXP}(\theta^{-1}) = [\mathrm{EXP}(\theta)]^{-1}, \qquad \mathrm{EXP}(\theta^{\dagger}) = [\mathrm{EXP}(\theta)]^{\dagger}. \tag{15}$$

2.3. Unitary operators

In the Hilbert space H we consider the unitary operator U

$$U = \exp\left[\mathrm{i}\sum_{M,K} a_M^{\dagger} \Lambda_{MK} a_K\right] \tag{16}$$

where Λ is a $d \times d$ Hermitian matrix. It is known (e.g. [6]) that

$$\begin{aligned} &U a_M U^{\dagger} = \sum_K V_{MK} a_K, \qquad && U a_M^{\dagger} U^{\dagger} = \sum_K V_{MK}^* a_K^{\dagger} \\ &V = \exp(-\mathrm{i}\Lambda), && V V^{\dagger} = \mathbf{1}_{\mathcal{G}}. \end{aligned} \tag{17}$$

The vacuum state remains invariant under these transformations. The total number of operator invariant under these transformations is

$$n_T \equiv \sum_M a_M^{\dagger} a_M = U\left[\sum_M a_M^{\dagger} a_M\right] U^{\dagger}. \tag{18}$$

We next consider a unitary operator θ acting on $\mathcal{G}$. It can be written as $\theta = \exp(\mathrm{i}\lambda\phi)$ where ϕ is a Hermitian operator and λ is a real number. Using equation (17) in conjunction with equation (14) we prove that

$$\mathrm{EXP}(\theta) = \exp\left[\sum_{M,K} a_M^{\dagger} (\ln\theta)_{MK} a_K\right] = \exp[\mathrm{i}\lambda\mathcal{U}_{\phi}], \qquad \mathcal{U}_{\phi} \equiv \sum_{M,K} a_M^{\dagger} \phi_{MK} a_K. \tag{19}$$

$\mathrm{EXP}(\theta)$ is a unitary operator, and $\mathcal{U}_{\phi}$ is a Hermitian operator. This shows that the general transformation of equation (16) in H is intimately connected through the exponential Hilbert formalism to the transformation $\theta = \exp(\Lambda)$ in $\mathcal{G}$.

It is easily seen that

$$[\mathcal{U}_{\phi}, \mathcal{U}_{\chi}] = \mathcal{U}_{[\phi,\chi]}. \tag{20}$$

In the special case that $\phi = \mathbf{1}_{\mathcal{G}}$ we get $\theta = \exp(\mathrm{i}\lambda \mathbf{1}_{\mathcal{G}}) = \mathrm{e}^{\mathrm{i}\lambda}\mathbf{1}_{\mathcal{G}}$ and

$$\mathrm{EXP}(\mathrm{e}^{\mathrm{i}\lambda}\mathbf{1}_{\mathcal{G}}) = \exp\left[\mathrm{i}\lambda \mathcal{U}_{\mathbf{1}}\right], \qquad \mathcal{U}_{\mathbf{1}} = n_T \tag{21}$$

$\mathcal{U}_{\mathbf{1}}$ is the total number of photons n_T.

More generally we assume that a family of operators θ acting on $\mathcal{G}$ form a Lie group $\mathcal{L}$ with Hermitian generators $\phi^{(1)}, \ldots, \phi^{(m)}$:

$$\theta = \prod_{\ell=1}^{m} \exp[\mathrm{i}\lambda_\ell \phi^{(\ell)}], \qquad [\phi^{(\ell)}, \phi^{(k)}] = c_{\ell k j}\phi^{(j)} \tag{22}$$

where λ_ℓ are real numbers and $c_{\ell k j}$ are the structure constants of the corresponding Lie algebra. We define the operators

$$\mathcal{U}_\ell = \sum_{M,K} a_M^\dagger \phi_{MK}^{(\ell)} a_K \tag{23}$$

and using equation (20) we show that they form the Lie algebra

$$[\mathcal{U}_\ell, \mathcal{U}_k] = c_{\ell k j}\mathcal{U}_j \tag{24}$$

and consequently the operators

$$\prod_{\ell=1}^{m} \mathrm{EXP}(\exp[\mathrm{i}\lambda_\ell \phi^{(\ell)}]) = \prod_{\ell=1}^{m} \exp[\mathrm{i}\lambda_\ell \mathcal{U}_\ell] \tag{25}$$

form a representation of the Lie group $\mathcal{L}$.

Equation (17) can now be rewritten as

$$\mathrm{EXP}(\theta) a_M\, [\mathrm{EXP}(\theta)]^\dagger = \sum_K a_K \theta_{KM}^*, \qquad \mathrm{EXP}(\theta) a_M^\dagger\, [\mathrm{EXP}(\theta)]^\dagger = \sum_K a_K^\dagger \theta_{KM}. \tag{26}$$

We next consider the subspace H_n spanned by all number states in H with total number of photons equal to n. The Hilbert space H is clearly the direct sum of all H_n. We call π_n the projection operator onto H_n. It is easily seen that

$$[\pi_n, \mathrm{EXP}(\theta)] = [\pi_n, \mathcal{U}_\phi] = 0. \tag{27}$$

Therefore these operators leave the spaces H_n invariant, i.e., acting on a state in H_n, produce another state also in H_n.

3. Displacements in $\mathcal{G}$ and the corresponding mode displacements in H

In this section we consider three special cases of the transformations of equation (16). They have been studied in [2]; but our purpose here is to express them as $\mathrm{EXP}(\theta)$ where θ is a transformation in $\mathcal{G}$.

3.1. Fourier transforms

The Fourier transform for modes U_F (see also [7]) is given by equation (16) with

$$\Lambda = \mathrm{i}\ln F, \qquad F_{MK} = d^{-1/2}\omega^{MK}, \qquad \omega = \exp\left(\frac{\mathrm{i}2\pi}{d}\right). \tag{28}$$

The $d \times d$ matrix F is the Fourier transform in the context of finite quantum systems (for a review see [8] and references therein). This leads to

$$U_F a_M U_F^\dagger = d^{-1/2}\sum_K \omega^{MK} a_K, \qquad U_F a_M^\dagger U_F^\dagger = d^{-1/2}\sum_K \omega^{-MK} a_K^\dagger. \tag{29}$$

In the context of the exponential Hilbert formalism U_F can be expressed as

$$U_F = \mathrm{EXP}(F^\dagger). \tag{30}$$

We next consider a vector z_M in $\mathcal{G}$ and with the finite Fourier transform F given in equation (28) we define the vector $\tilde{z}_K$:

$$\tilde{z}_K = \sum_{M=0}^{d-1} F_{KM} z_M. \tag{31}$$

Then equation (14) gives

$$\mathrm{EXP}(F^\dagger)\,\mathrm{EXP}[z_M] = \mathrm{EXP}[F^\dagger z_M] = \mathrm{EXP}\,[\tilde{z}_{-K}] \tag{32}$$

which can be rewritten as

$$U_F|\{z_M\}\rangle_{\mathrm{coh}} = |\{w_M\}\rangle_{\mathrm{coh}}, \qquad w_M = \sum_M F^*_{MK} z_K. \tag{33}$$

We note that

$$F^4 = \mathbf{1}_{\mathcal{G}} \to U_F^4 = \mathbf{1}_H. \tag{34}$$

3.2. Mode displacements

The mode displacement operator U_x is given by equation (16) with

$$\Lambda = \mathrm{i}\ln Z, \qquad Z_{MK} = \omega^M \delta(M, K). \tag{35}$$

In this case,

$$U_x = \exp\left[-\frac{\mathrm{i}2\pi}{d}\mathcal{U}_x\right], \qquad \mathcal{U}_x = \sum_{M,K} a_M^\dagger x_{MK} a_K, \qquad x_{MK} = M\delta(M, K). \tag{36}$$

The $d \times d$ matrices x and Z are position and displacements operators correspondingly in the context of finite quantum systems. $\mathcal{U}_x$ is the mode position operator. U_x is the exponential of $\mathcal{U}_x$ and performs mode displacements in the mode momentum direction.

The mode displacement operator U_p is given by equation (16) with

$$\Lambda = \mathrm{i}\ln X, \qquad X_{MK} = \delta(M, K+1). \tag{37}$$

In this case,

$$\begin{aligned} &U_p = \exp\left[\frac{\mathrm{i}2\pi}{d}\mathcal{U}_p\right], \qquad \mathcal{U}_p = \sum_{M,K} a_M^\dagger p_{MK} a_K = -U_F \mathcal{U}_x U_F^\dagger \\ &p_{MK} = \frac{d}{2\pi\mathrm{i}}\Delta_1(M-K). \end{aligned} \tag{38}$$

The $d \times d$ matrices p and X are momentum and displacements operators correspondingly in the context of finite quantum systems. The $\Delta_1(M-K)$ has been defined in [2, 8] and it is the discrete analogue of the derivative of delta function. $\mathcal{U}_p$ is the mode momentum operator. U_p is the exponential of $\mathcal{U}_p$ and performs mode displacements in the mode position direction. The operators U_x and U_p form a Heisenberg–Weyl group which has been discussed in [2].

In the context of the exponential Hilbert formalism U_x and U_p can be expressed as

$$U_x = \mathrm{EXP}(Z), \qquad U_p = \mathrm{EXP}(X^\dagger). \tag{39}$$

A vector z_M in $\mathcal{G}$ is displaced with the operators Z, X in the phase space $\mathcal{Z}_d \times \mathcal{Z}_d$ as follows:

$$X z_M = z_{M-1}, \qquad Z z_M = \omega^M z_M. \tag{40}$$

Here we use a short-hand notation where z_{M-1} is a vector whose M-component is equal to the $M-1$ component of the vector z_M (the integers belong in $\mathcal{Z}_d$, i.e., there is a cyclic structure). These displacements have been discussed in detail in [8]. A direct consequence of these relations (using equation (14)) is that

$$\begin{aligned}&\mathrm{EXP}(Z)\,\mathrm{EXP}[z_M]=\mathrm{EXP}[Zz_M]=\mathrm{EXP}[\omega^{-M}z_M]\\&\mathrm{EXP}(X^{\dagger})\,\mathrm{EXP}[z_M]=\mathrm{EXP}[X^{\dagger}z_M]=\mathrm{EXP}[z_{M+1}]\end{aligned}\tag{41}$$

which can be rewritten as

$$U_x|\{z_M\}\rangle_{\mathrm{coh}}=|\{z_M\omega^{-M}\}\rangle_{\mathrm{coh}},\qquad U_p|\{z_M\}\rangle_{\mathrm{coh}}=|\{z_{M+1}\}\rangle_{\mathrm{coh}}.\tag{42}$$

We note that

$$X^d=Z^d=\mathbf{1}_{\mathcal{G}}\rightarrow U_x^d=U_p^d=\mathbf{1}_H.\tag{43}$$

4. A continuum of modes

In the rest of the paper we consider a continuum of modes and the space $\mathcal{G}$ is infinite-dimensional and isomorphic to a harmonic oscillator Hilbert space. The vectors z_M with M in $\mathcal{Z}_d$ become now functions $z(x)$ with x in R. The space H is now a continuous tensor product. The usual procedure to convert the various relations from the discrete case to the continuous one is to take

$$x=(2\pi/d)^{1/2}M,\qquad p=(2\pi/d)^{1/2}K\tag{44}$$

and take the limit $d\rightarrow\infty$. Mode displacements, which previously took place in $\mathcal{Z}_d\times\mathcal{Z}_d$, will now take place in the $x-p$ mode phase space which is the plane $R\times R$.

In this case, the Fourier matrix F_{MK} of equation (28) becomes

$$F_{xy}=\frac{\Delta x}{(2\pi)^{1/2}}\,\mathrm{e}^{\mathrm{i}xy}.\tag{45}$$

The continuous creation and annihilation operators, denoted by $a^{\dagger}(x)$ and $a(x)$, are related to their discrete counterparts by

$$a_x^{\dagger}\rightarrow(\Delta x)^{1/2}a^{\dagger}(x),\qquad a_x\rightarrow(\Delta x)^{1/2}a(x),\qquad [a(x),a^{\dagger}(y)]=\delta(x-y).\tag{46}$$

The Hilbert space $\mathcal{G}$ is separable and it will be convenient to use a countable orthonormal basis, for example, the Hermitian basis

$$h_N(x)=\pi^{-1/4}2^{-N/2}(N!)^{-1/2}H_N(x)\exp\left[-\tfrac{1}{2}x^2\right].\tag{47}$$

In this case the function $z(x)$ is replaced with the sequence

$$z_N=\int\mathrm{d}x\,h_N(x)z(x).\tag{48}$$

We also introduce the creation and annihilation operators

$$a_N=\int\mathrm{d}x\,h_N(x)a(x),\qquad a_N^{\dagger}=\int\mathrm{d}x\,h_N(x)a^{\dagger}(x),\qquad \left[a_N,a_{N'}^{\dagger}\right]=\delta_{NN'}\tag{49}$$

which are labelled by the discrete variable N which takes the values $0,1,2,\ldots$. We refer to this as the 'N-representation', and to the one in equation (46) as the x-representation. The x-representation uses 'physical' creation and annihilation operators, which act separately on the physically distinct oscillators at the various points x. In this representation scalar products involve products of a non-countable set of variables. Such products have been defined by von Neumann [3], but they are not easy to use in practical calculations. For coherent states (and superpositions of a few coherent states) we give below equation (58), which can be used in practical calculations. The N-representation uses 'mathematical' creation and annihilation operators, each of which acts simultaneously on all oscillators in the continuum with weight $h_N(x)$. Its advantage is that N is a discrete variable and scalar products involve infinite products of a countable set of variables which are easier to use in practice. We note that the vacuum $|0\rangle \equiv |0, 0, \ldots\rangle$ is the same with respect to both the $a(x)$ and also the a_N operators.

The total number of photons is given by

$$n_T = \int \mathrm{d}x\, a^\dagger(x)a(x) = \sum_{N=0}^{\infty} a_N^\dagger a_N. \tag{50}$$

We emphasize the difference between the creation and annihilation operators $a_N, a_N^\dagger$ (or equivalently $a(x), a^\dagger(x)$) in H and 'creation and annihilation operators' in $\mathcal{G}$

$$c_{\mathcal{G}} = 2^{-1/2}[x + \mathrm{i}p], \qquad c_{\mathcal{G}}^\dagger = 2^{-1/2}[x - \mathrm{i}p], \qquad p = -\mathrm{i}\partial_x. \tag{51}$$

The former create and annihilate photons in a particular mode. The latter manipulate the distribution of photons into the various modes; they change the function $z(x)$ which through the EXP map describes amplitudes of coherent states in the various modes. Similarly the total number of photons given in equation (50) is a different concept from

$$N_{\mathcal{G}} = c_{\mathcal{G}}^\dagger c_{\mathcal{G}} = \tfrac{1}{2}\left(x^2 - \partial_x^2 - 1\right). \tag{52}$$

4.1. Coherent states

Displacement operators in the N- and x-representations are given by

$$\begin{aligned} D(\{z_N\}) &\equiv \exp\left[\sum_{N=0}^{\infty}\left(z_N a_N^\dagger - z_N^* a_N\right)\right] \\ D(\{z(x)\}) &\equiv \exp\left[\int \mathrm{d}x\,(z(x)a^\dagger(x) - z^*(x)a(x))\right]. \end{aligned} \tag{53}$$

We stress that these are displacements in the phase space of each oscillator, and it is a very different concept from the mode displacements in the $x - p$ mode phase space which we introduce later.

Coherent states are introduced as

$$|\{z_N\}\rangle_{\mathrm{coh}} = D(\{z_N\})|0\rangle \tag{54}$$

or equivalently as

$$|\{z(x)\}\rangle_{\mathrm{coh}} = D(\{z(x)\})|0\rangle = \exp\left[-\tfrac{1}{2}(z(x), z(x))\right]\mathrm{EXP}[z(x)] \tag{55}$$

where $(z(x), z(x))$ is the scalar product in $\mathcal{G}$ given in equation (3) with the summation replaced here by integration. It is clear that

$$z(x) = \sum_{N=0}^{\infty} z_N h_N(x) \rightarrow |\{z(x)\}\rangle_{\mathrm{coh}} = |\{z_N\}\rangle_{\mathrm{coh}}. \tag{56}$$

Using equation (50) we find that the total number of photons in these coherent states is

$$n_T = \int \mathrm{d}x |z(x)|^2 = \sum_{N=0}^{\infty} |z_N|^2. \tag{57}$$

The overlap of two coherent states is calculated using equation (5):

$$
{}_{\mathrm{coh}}\langle\{w(x)\}|\{z(x)\}\rangle_{\mathrm{coh}} = \exp\left[-\tfrac{1}{2}(w(x), w(x)) - \tfrac{1}{2}(z(x), z(x)) + (w(x), z(x))\right]. \tag{58}
$$

This is useful in practical calculations.

As an example we consider the case

$$
\begin{aligned}
&|\{z_N\}\rangle_{\mathrm{coh}} = |\{\zeta z_{\mathrm{gau}}(x;A)\}\rangle_{\mathrm{coh}}, \qquad z_N = \zeta \exp\left[-\frac{1}{2}|A|^2\right]\frac{A^N}{(N!)^{1/2}} \\
&z_{\mathrm{gau}}(x;A) = \pi^{-1/4}\exp\left[-\frac{1}{2}x^2 + 2^{1/2}Ax - AA_R\right]
\end{aligned} \tag{59}
$$

where $A = A_R + \mathrm{i}A_I$, and the complex factor ζ has been inserted so that the average number of photons is $|\zeta|^2$. They are special case of coherent states with amplitude which in the x-representation has Gaussian distribution among the various modes, and in the N-representation has 'square root of Poisson' distribution.

As a second example we consider the case

$$
|0, \ldots, 0, z_\Lambda, 0, \ldots\rangle_{\mathrm{coh}} = |\{\zeta h_\Lambda(x)\}\rangle_{\mathrm{coh}}, \qquad z_\Lambda = \zeta. \tag{60}
$$

In these coherent states the distribution of the amplitude among the various modes is the Hermitian $h_\Lambda(x)$ in the x-representation, and the $\zeta\delta(\Lambda, N)$ in the N-representation.

An important special case is when $A = 0$ in equation (59), which is equivalent to $z_N = \zeta\delta(N, 0)$ in equation (60). In this case

$$
|\zeta, 0, 0, \ldots\rangle_{\mathrm{coh}} = |\{\zeta z_{\mathrm{gau}}(x;0)\}\rangle_{\mathrm{coh}} = |\{\zeta h_0(x)\}\rangle_{\mathrm{coh}}. \tag{61}
$$

4.2. Fourier transform for modes

The Fourier transform for modes U_F is given by equation (30) where F is the Fourier transform in $\mathcal{G}$. Since $\mathcal{G}$ is here a harmonic oscillator Hilbert space, F is given by

$$
F = \exp\left[\mathrm{i}\frac{\pi}{2}N_{\mathcal{G}}\right]. \tag{62}
$$

Therefore,

$$
\begin{aligned}
&U_F = \mathrm{EXP}(F^\dagger) = \mathrm{EXP}\left[\exp\left(-\mathrm{i}\frac{\pi}{2}N_{\mathcal{G}}\right)\right] = \exp\left[-\mathrm{i}\frac{\pi}{2}\mathcal{U}_{N_{\mathcal{G}}}\right] \\
&\mathcal{U}_{N_{\mathcal{G}}} = \int \mathrm{d}x\, a^\dagger(x)N_{\mathcal{G}}a(x) = \frac{1}{2}\int \mathrm{d}x\, a^\dagger(x)\left(x^2 - \partial_x^2 - 1\right)a(x) = \sum_N N a_N^\dagger a_N.
\end{aligned} \tag{63}
$$

This leads to the relations

$$
\begin{aligned}
U_F a(x) U_F^\dagger &= (2\pi)^{-1/2}\int \mathrm{d}y\, a(y)\exp(\mathrm{i}xy) \\
U_F a^\dagger(x) U_F^\dagger &= (2\pi)^{-1/2}\int \mathrm{d}y\, a^\dagger(y)\exp(-\mathrm{i}xy)
\end{aligned} \tag{64}
$$

which are the analogues of equation (29), and also to the relations:

$$
U_F a_N U_F^\dagger = \mathrm{i}^N a_N, \qquad U_F a_N^\dagger U_F^\dagger = (-\mathrm{i})^N a_N^\dagger. \tag{65}
$$

5. Mode phase space

5.1. Mode position and mode momentum operators

We consider displacements acting on $\mathcal{G}$

$$d(\alpha,\beta,\gamma) = \exp(-\mathrm{i}\alpha x)\exp(\mathrm{i}\beta p)\exp(\mathrm{i}\gamma \mathbf{1}_{\mathcal{G}}) \tag{66}$$

We have included here the $\exp(\mathrm{i}\gamma \mathbf{1}_{\mathcal{G}})$ which simply gives a phase factor to a particular function. The mode displacements (acting on H) corresponding to each of these terms are

$$\begin{aligned}&\mathrm{EXP}[\exp(-\mathrm{i}\alpha x)] = \exp(-\mathrm{i}\alpha\mathcal{U}_x)\\ &\mathcal{U}_x = \int \mathrm{d}x\, x a^\dagger(x)a(x) = \sum_{N=0}^{\infty}\left(\frac{N+1}{2}\right)^{1/2}\left[a_{N+1}^\dagger a_N + a_N^\dagger a_{N+1}\right]\end{aligned} \tag{67}$$

and

$$\begin{aligned}&\mathrm{EXP}[\exp(\mathrm{i}\beta p)] = \exp(\mathrm{i}\beta\mathcal{U}_p)\\ &\mathcal{U}_p = -\mathrm{i}\int \mathrm{d}x\, a^\dagger(x)\partial_x a(x) = \mathrm{i}\sum_{N=0}^{\infty}\left(\frac{N+1}{2}\right)^{1/2}\left[a_{N+1}^\dagger a_N - a_N^\dagger a_{N+1}\right]\end{aligned} \tag{68}$$

and

$$\mathrm{EXP}[\exp(\mathrm{i}\gamma\mathbf{1}_{\mathcal{G}})] = \exp(\mathrm{i}\alpha\mathcal{U}_{\mathbf{1}}), \qquad \mathcal{U}_{\mathbf{1}} = \int \mathrm{d}x\, a^\dagger(x)a(x) = \sum_N a_N^\dagger a_N = n_T. \tag{69}$$

Taking into account equation (20) we show that

$$[\mathcal{U}_x,\mathcal{U}_p] = \mathrm{i}\mathcal{U}_{\mathbf{1}}, \qquad [\mathcal{U}_x,\mathcal{U}_{\mathbf{1}}] = [\mathcal{U}_p,\mathcal{U}_{\mathbf{1}}] = 0. \tag{70}$$

The $\mathcal{U}_x$ and $\mathcal{U}_p$ in equations (36), (38) are the discrete analogues of the $\mathcal{U}_x$ and $\mathcal{U}_p$ in equations (67), (68). In order to see this we need to take into account that the function $\Delta_1(M-K)$ in equation (38) is the discrete analogue of the derivative of delta function.

We next consider the subspace H_n spanned by all number states in H with total number of photons equal to n. The H_0 has only one state (the $|0,0,\ldots\rangle$), and all the other spaces H_n are infinite dimensional. We have explained earlier (equation (27)) that all operators $\mathcal{U}_\phi$ (which includes $\mathcal{U}_x,\mathcal{U}_p$) leave the spaces H_n invariant, i.e., acting on a state in H_n, produce another state also in H_n. Within a particular H_n equation (70) becomes

$$[\mathcal{U}_x\pi_n,\mathcal{U}_p\pi_n] = \mathrm{i}n\pi_n. \tag{71}$$

This can be viewed as a quantum mechanical commutation relation with strength of non-commutativity ('Planck constant') equal to n.

5.2. Mode uncertainties

We consider the operators

$$\begin{aligned}&\mathcal{U}_{x^2} = \int \mathrm{d}x\, x^2 a^\dagger(x)a(x), \qquad \mathcal{U}_{p^2} = -\int \mathrm{d}x\, a^\dagger(x)\partial_x^2 a(x)\\ &\mathcal{U}_{\frac{1}{2}(xp+px)} = \frac{1}{2}\int \mathrm{d}x[a^\dagger(x)x\partial_x a(x) + a^\dagger(x)\partial_x x a(x)].\end{aligned} \tag{72}$$

It is easily seen that

$$\mathcal{U}_{N_{\mathcal{G}}} = \tfrac{1}{2}[\mathcal{U}_{x^2} + \mathcal{U}_{p^2} - \mathcal{U}_{\mathbf{1}}]. \tag{73}$$

We next consider the coherent state $|\{z(x)\}\rangle_{\rm coh}$ of equation (55) and calculate the quantities:

$$\langle n_T\rangle = \int \mathrm{d}x |z(x)|^2, \qquad \langle \mathcal{U}_x\rangle = \int \mathrm{d}x\, x|z(x)|^2, \qquad \langle \mathcal{U}_{x^2}\rangle = \int \mathrm{d}x\, x^2|z(x)|^2. \tag{74}$$

We define the mode position uncertainty of this coherent state as

$$\Delta x = \left[\frac{\langle \mathcal{U}_{x^2}\rangle}{\langle n_T\rangle} - \left(\frac{\langle \mathcal{U}_x\rangle}{\langle n_T\rangle}\right)^2\right]^{1/2}. \tag{75}$$

In a similar way we define the Δp. The σ_{xp} of this coherent state is given by

$$\sigma_{xp} = \frac{\left\langle \mathcal{U}_{\frac{1}{2}(xp+px)}\right\rangle}{\langle n_T\rangle} - \frac{\langle \mathcal{U}_x\rangle\langle \mathcal{U}_p\rangle}{\langle n_T\rangle^2}. \tag{76}$$

It is clear that

$$(\Delta x\Delta p)^2 - \sigma_{xp}^2 \geqslant \tfrac{1}{4}. \tag{77}$$

Using the $(\langle \mathcal{U}_x\rangle, \langle \mathcal{U}_p\rangle)$ and also the uncertainties $\Delta x, \Delta p, \sigma_{xp}$ we can plot the uncertainty ellipse of the coherent state $|\{z(x)\}\rangle_{\rm coh}$ in the mode phase space $x - p$. The mode uncertainty ellipse indicates the location of the quantum state; in the sense that oscillators labelled with x outside it, are close to the vacuum state. Also the quantum state propagates in the chain of oscillators, with momenta inside the uncertainty ellipse. This will become more clear below when we study the time evolution.

General states are superpositions of many coherent states. In order to show how the above results are generalized to arbitrary states we consider for simplicity the following superposition of two coherent states

$$|s\rangle = \alpha|\{z(x)\}\rangle_{\rm coh} + \beta|\{w(x)\}\rangle_{\rm coh} \tag{78}$$

where α, β obey the normalization constraint:

$$|\alpha|^2 + |\beta|^2 + \exp\left[-\tfrac{1}{2}(z(x), z(x)) - \tfrac{1}{2}(w(x), w(x))\right][\alpha\beta^*\,\mathrm{e}^{(w(x),z(x))} + \alpha^*\beta\,\mathrm{e}^{(z(x),w(x))}] = 1. \tag{79}$$

We show that

$$\begin{aligned}
\langle s|n_T|s\rangle &= \int \mathrm{d}x|\alpha z(x) + \beta w(x)|^2 \\
\langle s|\mathcal{U}_x|s\rangle &= \int \mathrm{d}x\, x|\alpha z(x) + \beta w(x)|^2 \\
\langle s|\mathcal{U}_{x^2}|s\rangle &= \int \mathrm{d}x\, x^2|\alpha z(x) + \beta w(x)|^2.
\end{aligned} \tag{80}$$

It is seen that the state $|s\rangle$ has the same $(\langle \mathcal{U}_x\rangle, \langle \mathcal{U}_p\rangle)$ and the same mode uncertainties $\Delta x, \Delta p$ as the coherent state $|\alpha z(x) + \beta w(x)\}\rangle_{\rm coh}$; although of course the state $|s\rangle$ is not a coherent state and it is very different from the state $|\alpha z(x) + \beta w(x)\}\rangle_{\rm coh}$. These results are easily extended to superpositions of many coherent states, i.e., to general states.

As an example, we consider the coherent states $|\{\zeta z_{\rm gau}(x; A)\}\rangle_{\rm coh}$ of equation (59), and we get

$$\begin{aligned}
&\langle n_T\rangle = |\zeta|^2, \qquad \frac{\langle \mathcal{U}_x\rangle}{\langle n_T\rangle} = 2^{1/2}A_R, \qquad \frac{\langle \mathcal{U}_p\rangle}{\langle n_T\rangle} = 2^{1/2}A_I \\
&\Delta x = \Delta p = 2^{-1/2}, \qquad \sigma_{xp} = 0.
\end{aligned} \tag{81}$$

Another example is the coherent states $|\{\zeta h_\Lambda(x)\}\rangle_{\rm coh}$ of equation (60):

$$\langle n_T\rangle = |\zeta|^2, \qquad \langle \mathcal{U}_x\rangle = \langle \mathcal{U}_p\rangle = 0, \qquad \Delta x = \Delta p = \left(\Lambda + \tfrac{1}{2}\right)^{1/2}, \qquad \sigma_{xp} = 0. \tag{82}$$

5.3. General mode displacements

General mode displacements acting on H are given by

$$\mathcal{D}(\alpha, \beta, \gamma) \equiv \exp(-\mathrm{i}\alpha\mathcal{U}_x)\exp(\mathrm{i}\beta\mathcal{U}_p)\exp(\mathrm{i}\gamma\mathcal{U}_{\mathbf{1}}). \tag{83}$$

We note that the exponents are linear functions of $\mathcal{U}_x, \mathcal{U}_p$, i.e., they are quadratic functions of $a(x), a^\dagger(x)$. We show that

$$\exp(-\mathrm{i}\alpha\mathcal{U}_x)\exp(\mathrm{i}\beta\mathcal{U}_p) = \exp(\mathrm{i}\beta\mathcal{U}_p)\exp(-\mathrm{i}\alpha\mathcal{U}_x)\exp(\mathrm{i}\alpha\beta\mathcal{U}_{\mathbf{1}}) \tag{84}$$

and use this to prove

$$\mathcal{D}(\alpha_1, \beta_1, \gamma_1)\mathcal{D}(\alpha_2, \beta_2, \gamma_2) = \mathcal{D}(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2 - \beta_1\alpha_2). \tag{85}$$

A direct consequence of equation (26) is that

$$\begin{aligned}
\mathcal{D}(\alpha, \beta, \gamma)a(x)[\mathcal{D}(\alpha, \beta, \gamma)]^\dagger &= [d(\alpha, \beta, \gamma)]^\dagger a(x) \\
&= \exp[\mathrm{i}(\alpha x - \alpha\beta - \gamma)]a(x - \beta) \\
\mathcal{D}(\alpha, \beta, \gamma)a^\dagger(x)[\mathcal{D}(\alpha, \beta, \gamma)]^\dagger &= [[d(\alpha, \beta, \gamma)]^\dagger a(x)]^\dagger \\
&= \exp[-\mathrm{i}(\alpha x - \alpha\beta - \gamma)]a^\dagger(x - \beta).
\end{aligned} \tag{86}$$

The effect of mode displacements on $a(x)$ should be compared and contrasted with the effect of the displacements of equation (53) on these operators:

$$D(\{z(x)\})a(x)[D(\{z(x)\})]^\dagger = a(x) - z(x) \tag{87}$$

$$D(\{z(x)\})a^\dagger(x)[D(\{z(x)\})]^\dagger = a^\dagger(x) - z^*(x) \tag{88}$$

We next use equation (14) to show that the general coherent states of equation (55) are displaced as follows:

$$\mathcal{D}(\alpha, \beta, \gamma)|\{z(x)\}\rangle_{\mathrm{coh}} = |\{\mathrm{e}^{\mathrm{i}(\gamma - \alpha x)}z(x + \beta)\}\rangle_{\mathrm{coh}}. \tag{89}$$

This is the analogue of equations (42).

As an example, we consider the special case of coherent states given in equation (59) and using the general result of equation (14), we show that

$$\begin{aligned}
&\mathcal{D}(\alpha, \beta, \gamma)|\{\zeta z_{\mathrm{gau}}(x; A)\}\rangle_{\mathrm{coh}} = |\{\mathrm{e}^{\mathrm{i}\phi}\zeta z_{\mathrm{gau}}(x; A + B)\}\rangle_{\mathrm{coh}} \\
&B = -2^{-1/2}(\beta + \mathrm{i}\alpha), \qquad \phi = \gamma + \tfrac{1}{2}\alpha\beta + 2^{-1/2}(-\alpha A_R + \beta A_I).
\end{aligned} \tag{90}$$

Before the displacement, this state is located around the point $2^{1/2}A$ in the $x - p$ mode phase space viewed as a complex plane, and after the displacement, it is located around the point $2^{1/2}(A - B)$. These relations show again the difference between the mode displacements of equation (83) and the displacements of equation (53). Mode displacements are collective displacements of the full state in the chain of oscillators. The displacements of equation (53) displace the state of each oscillator within its own phase space.

5.4. Mode squeezing

We consider squeezing transformations acting on $\mathcal{G}$

$$\begin{aligned}
&s(r, \theta, \lambda) = \exp(\mathrm{i}r\sin\theta K_1 - \mathrm{i}r\cos\theta K_2)\exp(\mathrm{i}\lambda K_0) \\
&K_1 = \frac{1}{4}\left(x^2 + \partial_x^2\right), \qquad K_2 = \frac{\mathrm{i}}{4}(x\partial_x + \partial_x x), \qquad K_0 = \frac{1}{4}\left(x^2 - \partial_x^2\right)
\end{aligned} \tag{91}$$

where the K_0, K_1, K_2 obey the $SU(1, 1)$ algebra

$$[K_1, K_2] = -\mathrm{i}K_0, \qquad [K_2, K_0] = \mathrm{i}K_1, \qquad [K_0, K_1] = \mathrm{i}K_2. \tag{92}$$

Using them we define the following mode squeezing transformations acting on H:

$$\begin{aligned}&\mathrm{EXP}[\exp(\mathrm{i}r\sin\theta K_1 - \mathrm{i}r\cos\theta K_2)] = \exp\left(\mathrm{i}r\sin\theta\mathcal{U}_{K_1} - \mathrm{i}r\cos\theta\mathcal{U}_{K_2}\right)\\ &\mathcal{U}_{K_1} = \frac{1}{4}[\mathcal{U}_{x^2} - \mathcal{U}_{p^2}], \qquad \mathcal{U}_{K_2} = \frac{\mathrm{i}}{2}\mathcal{U}_{\frac{1}{2}(xp+px)}\end{aligned} \tag{93}$$

and

$$\begin{aligned}&\mathrm{EXP}[\exp(\mathrm{i}\lambda K_0)] = \exp\left(\mathrm{i}\lambda\mathcal{U}_{K_0}\right)\\ &\mathcal{U}_{K_0} = \tfrac{1}{4}[\mathcal{U}_{x^2} + \mathcal{U}_{p^2}] = \tfrac{1}{2}\mathcal{U}_{N_{\mathcal{G}}} + \tfrac{1}{4}\mathcal{U}_{\mathbf{1}}.\end{aligned} \tag{94}$$

Taking into account equation (20) we show that

$$\left[\mathcal{U}_{K_1}, \mathcal{U}_{K_2}\right] = -\mathrm{i}\mathcal{U}_{K_0}, \qquad \left[\mathcal{U}_{K_2}, \mathcal{U}_{K_0}\right] = \mathrm{i}\mathcal{U}_{K_1}, \qquad \left[\mathcal{U}_{K_0}, \mathcal{U}_{K_1}\right] = \mathrm{i}\mathcal{U}_{K_2}. \tag{95}$$

General mode squeezing transformations are defined as

$$\mathcal{S}(r,\theta,\lambda) = \exp\left(\mathrm{i}r\sin\theta\mathcal{U}_{K_1} - \mathrm{i}\cos\theta\mathcal{U}_{K_2}\right)\exp\left(\mathrm{i}\lambda\mathcal{U}_{K_0}\right). \tag{96}$$

A direct consequence of equation (26) is that

$$\begin{aligned}\mathcal{S}(r,\theta,\lambda)a(x)[\mathcal{S}(r,\theta,\lambda)]^\dagger &= [s(r,\theta,\lambda)]^\dagger a(x)\\ &= \int \mathcal{K}(x,y;r,\theta,\lambda)a(y)\,\mathrm{d}y\\ \mathcal{S}(r,\theta,\lambda)a^\dagger(x)\,[\mathcal{S}(r,\theta,\lambda)]^\dagger &= [[s(r,\theta,\lambda)]^\dagger a(x)]^\dagger\\ &= \int [\mathcal{K}(x,y;r,\theta,\lambda)]^* a^\dagger(y)\,\mathrm{d}y\end{aligned} \tag{97}$$

where $\mathcal{K}(x,y;r,\theta,\lambda)$ are the matrix elements $\langle x|s^\dagger(r,\theta,\lambda)|y\rangle$ in the space $\mathcal{G}$. The mode squeezing of $a(x)$ should be compared and contrasted with squeezing transformations on each individual oscillator. The latter are performed with the operators

$$\begin{aligned}S(\{r(x),\theta(x),\lambda(x)\}) = \exp&\left[\int \mathrm{d}x\left(-\frac{1}{4}r(x)\,\mathrm{e}^{-\mathrm{i}\theta(x)}a^\dagger(x)^2 + \frac{1}{4}r(x)\,\mathrm{e}^{\mathrm{i}\theta(x)}a(x)^2\right)\right]\\ &\times\exp\left[\mathrm{i}\int \mathrm{d}x\,\lambda(x)a^\dagger(x)a(x)\right]\end{aligned} \tag{98}$$

and they give

$$\begin{aligned}&Sa(x)S^\dagger = \mu(x)a(x) + \nu(x)a^\dagger(x)\\ &Sa^\dagger(x)S^\dagger = \mu^*(x)a^\dagger(x) + \nu^*(x)a(x)\\ &\mu(x) = \mathrm{e}^{-\mathrm{i}\lambda(x)}\cosh\left(\frac{r(x)}{2}\right), \qquad \nu(x) = \mathrm{e}^{-\mathrm{i}[\theta(x)+\lambda(x)]}\sinh\left(\frac{r(x)}{2}\right).\end{aligned} \tag{99}$$

As an example, we consider the special case of coherent states given in equation (59) and we show that

$$\mathcal{S}(r,\theta,\lambda)|\{\zeta z_{\mathrm{gau}}(x;A)\}\rangle_{\mathrm{coh}} = |\{\zeta z_{\mathrm{squ}}(x;A;r,\theta,\lambda)\}\rangle_{\mathrm{squ}} \tag{100}$$

where

$$\begin{aligned}&z_{\mathrm{squ}}(x;A;r,\theta,\lambda) = \epsilon_0\exp(-\epsilon_1x^2 + \epsilon_2x + \epsilon_3)\\ &\epsilon_0 = \pi^{-1/4}(\mu-\nu)^{-1/2}, \qquad \epsilon_1 = \frac{\mu+\nu}{2(\mu-\nu)}\\ &\epsilon_2 = \frac{2^{1/2}A}{\mu-\nu}, \qquad \epsilon_3 = -\frac{\mu^*-\nu^*}{\mu-\nu}\frac{A^2}{2} - \frac{|A|^2}{2}\\ &\mu = \mathrm{e}^{-\mathrm{i}\lambda}\cosh\left(\frac{r}{2}\right), \qquad \nu = \mathrm{e}^{-\mathrm{i}(\theta+\lambda)}\sinh\left(\frac{r}{2}\right).\end{aligned} \tag{101}$$

Mode squeezing changes the widths of the mode uncertainty ellipse. A quantum state is shared by the oscillators labelled with x inside the mode uncertainty ellipse, and with mode squeezing the set of these oscillators is reduced (or increased). At the same time the uncertainty in the momentum with which the state propagates is increased (or reduced).

6. Time evolution

An important property of coherent states is the 'temporal stability' [9]. This is the fact that coherent states evolve into other coherent states. Of course, this depends on the Hamiltonian of the system. In this section we consider the Hamiltonian

$$\mathcal{H}_1 = \mathcal{U}_{N_{\mathcal{G}}} \tag{102}$$

where $\mathcal{U}_{N_{\mathcal{G}}}$ has been given in equation (63). We show that the equations of motion are those of a harmonic oscillator, and that the general coherent states of equations (54), (55) have the property of temporal stability with respect to this Hamiltonian.

We prove that

$$\partial_t \mathcal{U}_x = \mathrm{i}[\mathcal{H}_1, \mathcal{U}_x] = \mathcal{U}_p, \qquad \partial_t \mathcal{U}_p = \mathrm{i}[\mathcal{H}_1, \mathcal{U}_p] = -\mathcal{U}_x. \tag{103}$$

They are equations of motion of a harmonic oscillator.

Acting on the general coherent states $|\{z_N\}\rangle_{\mathrm{coh}}$ of equation (54) we easily show that

$$\exp[\mathrm{i}t\mathcal{H}_1]|\{z_N\}\rangle_{\mathrm{coh}} = |\{z_N\,\mathrm{e}^{\mathrm{i}Nt}\}\rangle_{\mathrm{coh}}. \tag{104}$$

The same result can be expressed in the x-representation for the general coherent states of equation (55) as

$$\begin{gathered}\exp[\mathrm{i}t\mathcal{H}_1]|\{z(x)\}\rangle_{\mathrm{coh}} = |\{w(x)\}\rangle_{\mathrm{coh}} \\ w(x) = \int z(y)\mathcal{K}(y,x;t)\,\mathrm{d}y, \qquad \mathcal{K}(y,x;t) = \sum_N h_N(y)h_N(x)\,\mathrm{e}^{\mathrm{i}Nt}.\end{gathered} \tag{105}$$

In the special case of the coherent states of equation (59) we get

$$\exp[\mathrm{i}t\mathcal{H}_1]|\{\zeta z_{\mathrm{gau}}(x;A)\}\rangle_{\mathrm{coh}} = |\{\zeta z_{\mathrm{gau}}(x;A\,\mathrm{e}^{\mathrm{i}t})\}\rangle_{\mathrm{coh}}. \tag{106}$$

7. Location and propagation of entanglement

One application of our formalism is the study of entanglement location and propagation. Entanglement among d modes has been studied extensively in the last few years (e.g. [10–13]). There are many aspects of this problem. One is to quantify the amount of entanglement. Another is the distinction between separable and entangled mixed states, and the study of witness operators [14] which can distinguish between these two cases.

A lot of the early work on entanglement considered a finite number of qubits and more generally qudits. Continuous variable entanglement that involves a finite number of oscillators has later been studied [15]. Here we consider the next step which is a continuum of oscillators. We show how the operators $\mathcal{U}_x, \mathcal{U}_p$ and the mode uncertainty ellipse, which is related to them, can be used to quantify the location of entanglement, and the speed with which the entanglement propagates. We do not define a measure of entanglement; this can be a difficult task in a continuum of oscillators. However, this is not needed for our purposes.

Entanglement is considered with respect to a given tensor product factorization of the Hilbert space, i.e., with respect to a given set of subsystems. There are many ways of factorizing a Hilbert space as a tensor product of other spaces, i.e., of dividing a system into subsystems. It might be that one of them is 'physical' describing physically distinct subsystems; while the other ones are 'mathematical'. However it should be stressed that a state which is entangled with respect to one of these factorizations, might not be entangled with respect to another. As an example, we consider a simple number state in the N-representation

$$|1, 1, 0, \ldots\rangle_{\text{num}} = a_0^\dagger a_1^\dagger |0, 0, 0, \ldots\rangle. \tag{107}$$

This state is factorizable with respect to a tensor product factorization of the Hilbert space that involves the 'mathematical oscillators' described with $a_N, a_N^\dagger$. The same Hilbert space is the continuous tensor product of the 'physical oscillators' described with $a(x), a^\dagger(x)$, and taking into account equation (49) we see that with respect to them the state of equation (107) is entangled. We will use the term entanglement with respect to the latter 'physical factorization' that involves physically distinct oscillators at the various points x.

The mode uncertainty ellipse introduced earlier is applicable to all states; regardless of whether they are entangled or not. However if the states are entangled, it can be used to quantify the location of entanglement. In order to exemplify this we consider a system described with the Hamiltonian $\mathcal{H}_1$ of equation (102), which at $t = 0$ is in the entangled state

$$\begin{aligned} &|s(0)\rangle = \mathcal{N}[|\{\zeta_1 z_{\text{gau}}(x; A)\}\rangle_{\text{coh}} + |\{\zeta_2 z_{\text{gau}}(x; A)\}\rangle_{\text{coh}}] \\ &\mathcal{N} = \left[2 + 2\,\mathrm{e}^{-\frac{1}{2}|\zeta_1-\zeta_2|^2}\cos\phi\right]^{-1/2}, \qquad \phi = \Im(\zeta_1^*\zeta_2). \end{aligned} \tag{108}$$

Then using equation (106) we find that the state $|s(0)\rangle$ evolves as follows:

$$|s(t)\rangle = \mathcal{N}[|\{\zeta_1 z_{\text{gau}}(x; A\,\mathrm{e}^{\mathrm{i}t})\}\rangle_{\text{coh}} + |\{\zeta_2 z_{\text{gau}}(x; A\,\mathrm{e}^{\mathrm{i}t})\}\rangle_{\text{coh}}]. \tag{109}$$

Using equation (80) we find

$$\begin{aligned} &\langle n_T\rangle = \frac{|\zeta_1 + \zeta_2|^2}{2 + 2\,\mathrm{e}^{-\frac{1}{2}|\zeta_1-\zeta_2|^2}\cos\phi} \\ &\frac{\langle\mathcal{U}_x\rangle}{\langle n_T\rangle} = 2^{1/2}|A|\cos(\theta_A + t), \qquad \frac{\langle\mathcal{U}_p\rangle}{\langle n_T\rangle} = 2^{1/2}|A|\sin(\theta_A + t) \\ &\Delta x = \Delta p = 2^{-1/2}. \end{aligned} \tag{110}$$

The entanglement is located mostly within the interval $(\langle\mathcal{U}_x\rangle - \Delta x, \langle\mathcal{U}_x\rangle + \Delta x)$ which performs an oscillatory motion in time. Oscillators outside this region are close to the vacuum state. The average momentum of this motion of the entanglement is $\langle\mathcal{U}_p\rangle$ and its uncertainty is Δp.

Understanding of various aspects of entanglement in a continuum of modes is highly desirable since many branches of physics (e.g., particle physics and condensed matter) work primarily with a continuum of modes. We have introduced the concepts location and propagation of entanglement using a formalism based on the operators $\mathcal{U}_x, \mathcal{U}_p$. The next step is to introduce a measure of entanglement and it is highly desirable that this quantity should be conserved. Further work is required in this direction. Entanglement of a continuum of parties is an utmost interesting problem.

8. Discussion

The exponential Hilbert space approach provides an interesting link between quantum mechanics in a small d-dimensional Hilbert space $\mathcal{G}$ and quantum mechanics in a large Hilbert space H comprised of d coupled oscillators. Transformations in the small Hilbert space $\mathcal{G}$ lead naturally to collective transformations in H that involve all oscillators.

Using this approach, we have considered a continuum of coupled oscillators ($d \to \infty$) and introduced the mode position and mode momentum operators whose expectation values $\langle \mathcal{U}_x \rangle$ and $\langle \mathcal{U}_p \rangle$ describe the average position of the quantum state in the line of oscillators, and also the average momentum with which it propagates. They obey the commutation relation of equation (70) and a consequence of this is the uncertainty relation of equation (77). Oscillators outside the region $(\langle \mathcal{U}_x \rangle - \Delta x, \langle \mathcal{U}_x \rangle + \Delta x)$ are close to the vacuum state, and also the propagation of the quantum state in the chain of oscillators occurs with momenta in the region $(\langle \mathcal{U}_p \rangle - \Delta p, \langle \mathcal{U}_p \rangle + \Delta p)$.

The exponentials of $\mathcal{U}_x$ and $\mathcal{U}_p$ are mode displacement operators. Acting on a state they translate it in the chain of oscillators, and they also change its mode momentum. We also introduced the operators $\mathcal{U}_{x^2}$ and $\mathcal{U}_{p^2}$ and their exponentials which are mode squeezing operators. Acting on a quantum state, they squeeze its mode uncertainty ellipse.

Assuming that the Hamiltonian of the system is $\mathcal{H}_1$ in equation (102), we have shown in equation (103) that the equation of motion for $\mathcal{U}_x$ and $\mathcal{U}_p$ are similar to those of a harmonic oscillator. Consequently the coherent states evolve as described in equations (104), (105), (106).

In the case of entangled states, the formalism can be used to quantify the location of entanglement and the speed with which the entanglement propagates. For example, in the case considered in the previous section the entanglement is located mostly within the interval $(\langle \mathcal{U}_x \rangle - \Delta x, \langle \mathcal{U}_x \rangle + \Delta x)$ and oscillates in time.

We have considered one-dimensional continuum of oscillators, but generalizations to two and three dimensions are possible. The work can be used for the description of quantum phenomena in chains of coupled oscillators.

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